HYPER VERSION

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# Consistency conditions for holographic duality

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ABSTRACT: We show that if the beta functions of a field theory are given by the gradient of a certain potential on the space of couplings, a gravitational background in one more dimension can express the renormalization group (RG) flow of the theory. The field theory beta functions and the gradient flow constraint together reconstruct the second-order spacetime equations of motion. The RG equation reduces to the conventional gravitational computation of the spacetime quasilocal stress tensor, and a c-theorem holds true as a consequence of the Raychaudhuri equation. Conversely, under certain conditions, if the RG evolution of a field theory possesses a monotonic c-function, the flow of couplings can be expressed in terms of a higher-dimensional gravitational background.

KEYWORDS: Renormalization Regularization and Renormalons, AdS-CFT Correspondance.

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## Contents



## 1. Introduction

The holographic principle states that the degrees of freedom describing quantum gravity in some volume can be encoded at fixed density on a "screen" or surface that bounds that volume [1, 2]. A particularly simple realization of this principle appears to occur for asymptotically anti-de Sitter (AdS) spaces where $(d+1)$-dimensional spacetime dynamics is conjectured to be encoded in a local quantum field theory (QFT) on the timelike, $d$-dimensional AdS boundary [3].

In this correspondence, phenomena occuring closer to the AdS boundary are related to local, ultraviolet physics in the field theory, while infrared and non-local data encode the deep interior of the space [4]–[10]. This suggests that the semi-classical structure called spacetime can sometimes be generated from a QFT via renormalization group (RG) flow.[1]

Here, we show that if the beta functions of a field theory in 4 dimensions are given by the gradient of a certain potential on the space of couplings, the RG flow of the theory admits a description in terms of 5-dimensional gravity coupled to scalar fields. For such flows, the second-order spacetime equations of motion can be reconstructed from the field theory beta functions and the gradient flow constraint. The field theory RG equation is realized as the conventional gravitational computation of the trace of the quasilocal stress tensor. A c-theorem is satisfied by the RG flow as a consequence of the Raychaudhuri equation for the 5-dimensional gravitational background. Conversely, if a QFT has a c-function that evolves monotonically down an RG trajectory, under certain conditions the flow of couplings may be expressed in terms of a 5-dimensional gravitational background.

[1] In fact, it is known that the string field theory equations of motion are closely related to the Wilsonian RG flow of the string sigma model [11].



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## 2. Gradient beta functions

Consider a 4-dimensional QFT on a Ricci-flat manifold with an interaction la grangian:

$$S_{\text{int}} = \int d^4x\, \phi^I O_I\,. \tag{2.1}$$

On a flat manifold there is no conformal anomaly, and the RG equation is simply given by the conformal Ward identity for the trace of the stress tensor:

$$\langle T\rangle \equiv \langle T_i^i\rangle \propto \frac{d\,\Gamma}{d\log\lambda} = \beta^I \frac{\partial\Gamma}{\partial\phi^I}\,. \tag{2.2}$$

$\Gamma$ is quantum effective action of the field theory, the beta functions are the scale derivatives of the couplings $\beta^I = d\phi^I/d\log\lambda$, and $\lambda = \Lambda/\Lambda_0$ in terms of an energy cutoff $\Lambda$ and a reference scale $\Lambda_0$.

Now consider a QFT in which the beta fuctions can be derived as gradients of a particular potential on the couplings:

$$\beta^I = -G^{IJ}(\phi)\frac{\partial}{\partial\phi^J}\log(a\langle T\rangle + U(\phi)), \tag{2.3}$$

$G^{IJ}$ is a positive, symmetric function of the couplings $\phi^I$, $a$ is a constant of length dimension 4, and $U(\phi)$ is a function of couplings that will be related to the norm of the beta functions. We will show that near a conformal point the inverse metric $G_{IJ}$ must be related to the normalization of the 2-point function $\langle O_I O_J\rangle$. At a conformal point the beta functions vanish. Then (2.3) implies that $\partial(\langle T\rangle + U)/\partial\phi^I$ also vanishes, so that $(\langle T\rangle + U)$ is extremized as a function of the couplings.

The QFT RG flow can be related to the equations of motion of 5d gravity if, in addition, the norm of the beta functions is:

$$\frac{1}{4} G_{IJ}\,\beta^I\,\beta^J = 1 - \frac{V(\phi)}{(a\,\langle T\rangle + U(\phi))^2} \geq 0\,. \tag{2.4}$$

$V(\phi)$ will be related to a c-function for the RG flow. The ultraviolet limit of the theory, if it exists, is conformal. In this limit, $\langle T\rangle \to 0$, and we can choose the normalization $U(\phi)\to 1$. Furthermore, at *any* conformal point, the vanishing of the beta functions requires that $V(\phi) \to \langle T\rangle + U(\phi)$, implying that $V(\phi)$ is extremized. We will show that the first-order RG flow of the $d$-dimensional QFT can be represented as the second-order equations of motion of 5-dimensional gravity coupled to scalars $\phi^I$, with a potential $V(\phi)$ and a "boundary" cosmological constant related to $U(\phi)$.

At first glance, there are three unknowns in (2.3) and (2.4): the metric $G^{IJ}$ and the potentials $U(\phi)$ and $V(\phi)$. Since there are only two equations, this suggests that *any* RG flow is expressible in this manner by a suitable choice of metric and potentials. Near a conformal point, we will show that these quantities relate to other dynamical data such as correlation functions. But away from a conformal point, despite the constraints implied by the positivity and symmetry of the metric, (2.3) and (2.4) seem to be rather weak restrictions. However, we will argue that any flow that can be written as (2.3) and (2.4) possesses a c-function related to the potentials $U$ and $V$, provided a certain positive energy condition is satisfied. The inequality in (2.4) will be equivalent to a statement that field theories obeying (2.3) have a c-function that decreases monotonically during RG flow. So we will be forced to conclude either that a surprisingly general class of field theories possesses a monotonic c-function, or that the conditions (2.3) and (2.4) are much more restrictive than they appear.

## 3. Gravitational description

Einstein gravity coupled to scalars on a 5-dimensional manifold $\mathcal{M}$ has an action

$$S_h = \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^4x\,dr\,\sqrt{-h}\left[-R_h + \frac{1}{2}g_{IJ}\,h^{\mu\nu}\,\left(\partial_\mu\alpha^I\right)\left(\partial_\nu\alpha^J\right) - \frac{12}{\ell^2}\,v(\alpha)\right] - \\ -\frac{1}{8\pi G_N}\int_{\partial\mathcal{M}} d^4x\,\sqrt{\gamma}\,[\Theta + L_{\text{c.t.}}] \tag{3.1}$$

$R_h$ is the Ricci scalar of the spacetime metric $h_{\mu\nu}$, $g_{IJ}$ is the metric on the space of scalars $\alpha^I$ and $\ell$ is a length scale. By choice, the potential $v$ has negative extrema, with at least one extremum at $v=-1$. Placing the scalars at these points induces a negative cosmological constant in the space. We will be interested in solutions whose potentials approach $v=-1$ as $r\to\infty$. The boundary extrinsic curvature $\theta$ makes the equations of motion well defined, and $L_{\text{c.t.}}$ is a counterterm lagrangian constructed from intrinsic invariants of the induced metric $\gamma_{ij}$ on the spacetime boundary. When the scalars are at the $v(\alpha)=-1$ extremum, the gravitational part of the action can be rendered finite by setting $L_{\text{c.t.}} = \frac{3}{\ell}(1-\frac{\ell^2}{12}R_\gamma)$ [12, 13]. We require a counterterm scheme that yields a finite gravitational action for any solution that asymptotically approaches an extremum of the scalar potential. One such scheme is

$$L_{\text{c.t.}} = \frac{3}{\ell}\left(u(\alpha) - \frac{\ell^2}{12\,u(\alpha)}R_\gamma\right), \tag{3.2}$$

subject to the requirement that $u(\alpha)^2 \to -v(\alpha)$ when the $\alpha^I$ approach any extremum of $v$.[2] In effect, $u(\alpha)$ serves as a cosmological constant on the boundary of the space. Similarly, counterterms may be added to cancel divergences in the total action for the scalar fields [15].[3]

[2]The arguments of $u(\alpha)$ are the boundary values of the scalars. Also see [14] for discussions for boundary counterterms in theories with scalars.

[3]As we will discuss later, there is a large scheme dependence in this counterterm prescription, in parallel with scheme dependences in field theory RG flows.



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The most general 4-dimensional Poincaré invariant solution of this action can be put in the form

$$ds^2 = e^{2A(r)}\,\eta_{ij}\,dx^i dx^j + dr^2 \tag{3.3}$$

with $\eta_{ij}$ the flat metric in 4 dimensions and the scalars $\phi$ chosen as functions of $r$ only. The second-order equations of motion that follow from varying (3.1) with respect to the metric and requiring a solution of the form (3.3) are [16]:

$$\frac{d^2A}{dr^2} = -\frac{1}{6}\,g_{IJ}\,\frac{d\alpha^I}{dr}\,\frac{d\alpha^J}{dr}\,, \tag{3.4}$$

$$\left(\frac{dA}{dr}\right)^2 = -\frac{1}{\ell^2}\,v(\alpha) + \frac{1}{24}\,g_{IJ}\,\frac{d\alpha^I}{dr}\,\frac{d\alpha^J}{dr}\,. \tag{3.5}$$

The second equation is simply the statement of $r$-reparametrization invariance of the action (3.1) — i.e. it is the hamiltonian constraint. The scalar equations of motion are

$$g_{KL}\,\frac{d^2\alpha^L}{dr^2} + 4\,g_{KL}\,\frac{d\alpha^L}{dr}\,\frac{dA}{dr} = \frac{12}{\ell^2}\,\frac{\partial v}{\partial\alpha^K} + \frac{1}{2}\frac{\partial g_{IJ}}{\partial\alpha^K}\,\frac{d\alpha^I}{dr}\,\frac{d\alpha^J}{dr} - \frac{\partial g_{KL}}{\partial\alpha^I}\,\frac{d\alpha^I}{dr}\,\frac{d\alpha^L}{dr}\,. \tag{3.6}$$

So long as $d\alpha^I/dr \neq 0$, we can treat $A' \equiv dA/dr$ and $\alpha^{I'} \equiv d\alpha^I/dr$ as functions of $\alpha$, to write (3.4) as[4]

$$\frac{\partial A'}{\partial\alpha^I}\,\frac{d\alpha^I}{dr} = -\frac{1}{6}g_{IJ}\,\frac{d\alpha^I}{dr}\,\frac{d\alpha^J}{dr}\,. \tag{3.7}$$

Solutions to (3.7) are obtained by setting

$$\frac{d\alpha^I}{dr} = -6\,g^{IJ}\,\frac{\partial A'}{\partial\alpha^J}\,. \tag{3.8}$$

Computing $A'(\alpha)$ from (3.8) and (3.5), it is easy to show that the scalar eqs. (3.6) are automatically satisfied.[5] Thus, starting from (3.8) and (3.5) one can derive the complete equations of motions for a 5-dimensional lagrangian of the form (3.1).

Integrating the solutions for $\phi^{I'}$ and $A'$ yields a trajectory $\alpha^I(r)$ in the $N$-dimensional space of scalar fields. The $N$ first-order equations (3.7) and the the $N$-dimensional first-order equation (3.5) produce $2N$ integration constants [16]. Along with the specification of the integration bound $r_0$ and $A(r_0)$, these constants reproduce the $2N+2$ expected initial conditions of the scalar and gravity equations of motion [16].

Now consider the total action as a functional of the induced "boundary" metric $\gamma_{ij}$ on a surface of fixed $r$. Following Brown and York [17], and including the counterterms (3.2) [13], the quasilocal stress tensor of the spacetime on a fixed-$r$ surface

[4] The analysis below will always hold piecewise in domains where $d\alpha^I/dr \neq 0$. When the scalars are at an extremum of $v$, $\alpha^{I'} = 0$ and $A'$ is a constant. This will correspond to a conformal point in the field theory.

[5] To show this, differentiate (3.8) with respect to $r$ and use the $\alpha^I$ derivative of (3.5).

is the response of the action to variations of $\gamma_{ij}$:

$$\tau_{ij} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_h}{\delta \gamma^{ij}} = \frac{1}{8\pi G_N} \left[ \theta_{ij} - \theta\, \gamma_{ij} - \frac{3\, u(\alpha)}{\ell} \gamma_{ij} - \frac{\ell}{2\, u(\alpha)} \mathcal{G}_{ij} \right]. \tag{3.9}$$

Here $\theta_{ij}$ is the extrinsic curvature of the fixed-$r$ surface, $\theta$ is the trace of $\theta_{ij}$, and $\mathcal{G}_{ij}$ is the Einstein tensor constructed from the boundary metric. From the Hamilton-Jacobi perspective, $\tau_{ij}$ is simply the variable conjugate to the boundary metric $\gamma_{ij}$, given the action (3.1) as a functional of boundary data [17].

For solutions of the form (3.3), the trace of $\tau_{ij}$ is

$$\tau \equiv \gamma^{ij}\, \tau_{ij} = \frac{3}{2\pi G_N \ell} \left[ \ell \frac{dA}{dr} - u(\alpha) \right]. \tag{3.10}$$

We found solutions to the equations of motion by assuming $d\alpha^I/dr \neq 0$, so that $A'$ and $\alpha^{I'}$ could be expressed as functions of $\alpha^I$. Under these conditions, the trace of the Hamilton-Jacobi expression for the Brown-York stress tensor is proportional to

$$\frac{\delta S_h}{\delta A} = \frac{\delta S_h}{\delta \alpha^I} \frac{\delta \alpha^I}{\delta A}. \tag{3.11}$$

(Recall that $\tau = (D/\sqrt{-\gamma})\,(\delta S/\delta A)$.)

In the connection between CFTs and gravity on AdS spaces, position in the radial direction is known to play the role of the scale in dual field theory computations. It is difficult to map the radial coordinate directly into field theory scales because of the possibility of radial reparametrization. However, there is an invariant way to study radial positions in classical solutions of the form (3.3) — we will associate the warp factor in the metric ($A$), with the log of $\lambda$, the scale in the definition of the beta functions in (2.2). The map between the field theory RG equations and the gravitational equations of motion is then:

$$\begin{array}{llll} G^{IJ} \equiv 6\, g^{IJ} & \phi^I \equiv \alpha^I & a \equiv 2\,\pi\, G_N \dfrac{\ell}{3} & \log \lambda = A \\ \langle T \rangle \equiv \tau & V(\phi) \equiv -v(\alpha) & U(\phi) \equiv u(\alpha) & \end{array} \tag{3.12}$$

Using these substitutions and the fact that $A' = (a\,\tau + u(\alpha))/\ell$ (3.10), it easy to show that the equations of motion (3.8) and (3.5) are exactly the gradient beta function equation (2.3) and the potential equation (2.4), respectively. As discussed, simultaneous solution of (3.8) and (3.5) is sufficient to solve the complete coupled second-order equations for the scalars and the spacetime metric subject to the 4d Poincaré invariant ansatz (3.3). Identifying variations of $\Gamma$, the QFT effective action as a function of couplings, with variations of $S$, the spacetime action as a function of boundary data, the expression for the trace of Brown-York stress tensor (3.11) becomes equivalent to the field theory RG equation (2.2).

We have mapped the boundary values of the spacetime scalar fields and the metric to field theory couplings. That accounts for half the integration constants of the equations of motion (3.4)–(3.6). The remaining constants are the radial derivatives of bulk fields which, in the AdS/CFT context, were related to expectation values of operators in the CFT [6]. A similar relationship holds here; for example, $\langle T\rangle = \tau \sim dA/dr$ according to (3.10). Our analysis treats RG flow in the vacuum state where operator expectation values have been set to zero. From the gravitational perspective, this should correspond to fixing the radial derivatives of bulk fields to yield the lowest energy given the boundary values.

In perturbative string theory, a central consistency condition is the requirement of world-sheet conformal invariance, which implies that strings can consistently propagate only in backgrounds which satisfy the string equation of motion. In other words, the condition that the beta function is zero is equivalent to the target space equation of motion [18]. Equation (2.3) is an analogue of this statement in the context of holographic duality. Supplemented by the requirement (2.4) on the norm of the beta functions, (2.3) yields the full equations of motion of the spacetime fields. As we shall see, the potential in the gradient beta function equation is intimately related to a c-function which decreases monotonically along our RG flows. After the identifications above, (2.4) became the condition of $r$-reparametrization invariance of the 5-dimensional spacetime. This suggests that it should be directly related to invariance of the field theory under redefinitions of the floating scale.

## 4. C-functions and an analogue Zamolodchikov metric

Any field theory satisfying the constraint (2.3) has a c-function that decreases monotonically during RG-flow. The basic reason for this is that $\langle T\rangle$ is identified with the trace of $\tau_{ij}$, the quasilocal stress tensor of spacetime given in (3.9). Since the intrinsic curvatures of the boundaries at fixed $r$ vanish for metrics of the form (3.3), the trace of the stress tensor depends only on the extrinsic curvature:

$$\langle T\rangle \equiv \tau = -\frac{3}{8\pi G}\left(\theta + \frac{4u(\alpha)}{\ell}\right). \tag{4.1}$$

The Raychaudhuri equation implies a monotonic radial flow of $\theta$,

$$\frac{d\theta}{dr} \leq 0\,, \tag{4.2}$$

so long as a form of the weak positive energy condition is satisfied by the scalar fields [19].[6]

[6]Monotonicity follows from the Raychaudhuri equation describing evolution of the expansion $\theta$ along null curves, applied to static metrics of the form (3.3). If $\mathcal{T}_{ab}$ is the matter stress tensor and $\mathcal{T}$ is its trace, the required energy condition is that $\mathcal{T}_{ab}\,k^a\,k^b \geq 0$ for all null $k^a$. In [19], the weak energy condition is defined with respect to timelike $\xi^a$: $\mathcal{T}_{ab}\,\xi^a\,\xi^b \geq 0$. Both this, and the strong positive energy condition in [19], $\mathcal{T}_{ab}\,\xi^a\,\xi^b \geq -\frac{1}{2}\mathcal{T}$, imply the positivity of $\mathcal{T}_{ab}\,k^a\,k^b$.



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This fact, coupled with dimensional analysis and Bousso's covariant entropy formula [2], has led Sahakian [20] to propose a candidate gravitational analogue of a c-function:[7]

$$c \propto \frac{1}{G_N\, \theta^3}\,. \tag{4.3}$$

(See [21]–[24] for interesting examples and related proposals.) Translating into field theory variables, the candidate c-function is

$$c = c_0 \left(a \,\langle T\rangle + U(\phi)\right)^{-3}\,; \quad c_0 = \frac{b\,\ell^3}{G_N} \tag{4.4}$$

with $b$ a dimensionless numerical constant. The Raychaudhuri equation automatically implies monotonicity of this c-function. Accepting this proposal, the gradient beta function equation (2.3) may be rewritten as

$$\beta^I = \frac{1}{3}\, G^{IJ}\, \frac{\partial \log c}{\partial \phi^J} \equiv \tilde{G}^{IJ}\, \frac{\partial c}{\partial \phi^J}\,. \tag{4.5}$$

This formula gives a useful consistency check on the identifications we have been performing between field theory and gravitational quantities. We will show that $\tilde{G}^{IJ} = G^{IJ}/3c$ determines the normalization of the field theory 2-point function in the vicinity of a conformal point.

To see this, recall that perturbing around a conformal point by some marginal operators $O_I$, an analogue of the c-function in two dimensions can be defined as

$$T(x) \propto \beta^I O_I\,; \qquad \langle T(k)\, T(-k)\rangle = \tilde{c}\, k^4 \tag{4.6}$$

in terms of the trace of the stress tensor $T = T_i^i$. At a conformal point, the 2-point function of $O_I$ is given by $\langle O_I(x)\, O_J(0)\rangle = G^Z_{IJ}/x^8$, so that in the vicinity of this point

$$\langle O_I(k) O_J(-k)\rangle = k^4 \left(\text{const.} + G^Z_{IJ} \log\frac{k}{\Lambda_0} + \cdots\right) \equiv G^Z_{IJ}\, f(k) + \cdots\,. \tag{4.7}$$

Here $G^Z_{IJ}$ is an analogue of the Zamolodchikov metric defined in two dimensions. The last two equations together imply that

$$\tilde{c} = \text{const.} + \beta^I \beta^J G^Z_{IJ} \log\frac{k}{\Lambda_0} + \cdots\,. \tag{4.8}$$

An alternative expression for $c$ can be obtained by expanding the couplings around the conformal point as $\phi^I = \phi^I_0 + \beta^I \log\frac{k}{\Lambda_0} + \cdots$ and expanding $c(\phi)$ around $\phi_0$:

$$\tilde{c} = \text{const.} + \frac{\partial c}{\partial \phi^I} \beta^I\, \log\frac{k}{\Lambda_0} + \cdots\,. \tag{4.9}$$

[7] We are specializing the proposal to metrics of the form (3.3).



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The two expressions for $\tilde{c}$ are equal close to a conformal point if

$$\beta^I = G_Z^{IJ} \frac{\partial \tilde{c}}{\partial \phi^J}\,, \tag{4.10}$$

$G_Z^{IJ}$ being the inverse of $G_{IJ}^Z$. In two dimensions, this formula is exactly true by Zamolodchikov's theorem [25], while the above is an approximate derivation in four dimensions, in the vicinity of a conformal point. In fact, various candidates have been proposed for a c-function in 4-dimensional field theory [26]. The coefficient $\tilde{c}$ in (4.6) is one of these, and is related to the coefficient of the Weyl tensor squared term in the conformal anomaly [27, 28]. Another candidate is the coefficient of the Euler invariant in the anomaly. The material point for us is that eq. (4.10) holds true near a conformal point for either candidate c-function (see [27, 28, 29] and references therein).

Comparing (4.5) and (4.10) shows that consistency of the identification of the c-function in the former requires that $\tilde{G}_{IJ}$ determine the 2-point function of $O_I$ at the conformal point. At the UV conformal point, the beta functions and $\langle T\rangle$ are zero, while $u(\phi) = 1$; we then expect that $\langle O_I(k)O_J(-k)\rangle = f(k)\tilde{G}_{IJ} = 3c\, f(k)G_{IJ}$. This serves as a consistency check on the dictionary between field theory and gravity quantities that we are developing. We equated the spacetime action $S$, as a functional of boundary data, to the field theory quantum effective action $\Gamma$, as a functional of couplings. So the field theory 2-point function relates directly to the spacetime propagator between boundary points for the scalars $\alpha_I$.[8] We would like to see that this is proportional to $f(k)G_{IJ}(G_N/\ell^3)$.

At the ultraviolet conformal point, the potential $v(\alpha)$ in (3.1) was normalized to $-1$, and the equations of motion are solved to give pure anti-de Sitter space. So the scalar propagator in spacetime is given by the standard computation in the AdS/CFT correspondence [3, 30, 31] and is proportional in momentum space to $f(k) = k^4 \log k$. The normalization of the scalar fields in (3.1) implies that the scalar propagator for $\alpha_I$ is proportional to $G_N\, g_{IJ}$. Since $G_N$ has length dimension three and $\ell$ is only remaning length scale, the scalar propagator must yield, via the QFT-gravity dictionary,

$$\langle O_I\, O_J\rangle \propto f(k)\, G_{IJ}\, \frac{G_N}{\ell^3} + \cdots \tag{4.11}$$

in the vicinity of the UV conformal point. In the identification of the RG equations and the spacetime equations of motion, only the combination of parameters $a = 2\pi G_N \ell/3$ appeared. Now we see that this data can combine with the normalization of the 2-point function of $O_I$ to separately yield the Newton constant ($G_N$) and the spacetime curvature scale ($\ell$). Analysis of higher point functions would give further consistency conditions.

[8] Note the lower index.

The results we have accumulated suffice to show that the RG flow of any theory with a monotonic c-function can be be rewritten in terms of a 5d gravity background under some conditions. Suppose we are given as data the beta functions, stress tensor and monotonic c-function of a 4-dimensional QFT. The relation (4.5) serves to define a metric $G^{IJ}$, which we require to be positive and symmetric. Equation (4.4) defines the potential $U(\phi)$, and thereby the gradient beta function equation (2.3). Finally, consider the scale variation of the c-function:

$$\frac{d \log c}{d \log \lambda} = \beta^I \frac{\partial \log c}{\partial \phi^I} \,. \tag{4.12}$$

Inverting (4.5) to get an equation for $\partial \log c / \partial \phi^J$, we find:

$$\frac{d \log c}{d \log \lambda} \propto -G_{IJ}\, \beta^I\, \beta^J \,. \tag{4.13}$$

The right-hand side of this equation defines the potential $V(\phi)$ in (2.4). We now see that $V(\phi)$ is related to the deviation of the c-function away from its value at the conformal point. In summary, given the beta functions, stress tensor and monotonic c-function of a 4d QFT, we have defined a metric and two potentials. As we have shown, these are the elements of a 5-dimensional gravitational lagrangian whose equations of motion reproduce the field theory RG equations.

## 5. Discussion

The hamiltonian constraint (3.5) expresses radial reparametrization invariance of solutions to the action (3.1). If radial positions are mapped to field theory scales, we would expect to extract a "holographic" RG equation from this constraint. In [32] de Boer, Verlinde and Verlinde achieve this goal by separating the 5-dimensional bulk spacetime action $S_h$ bounded at a given radial position into one piece which is local in the boundary data ($S_{\mathrm{l}}$), and another which is non local ($S_{\mathrm{nl}}$). The hamiltonian constraint, i.e. the Hamilton-Jacobi equation for the total spacetime action $S_h$, is rewritten as a first-order RG equation of the Callan-Symanzik type for the non-local action $S_{\mathrm{nl}}$. The beta-functions are implicitly determined by the form of the local action $S_{\mathrm{l}}$ since they are defined as ratios of $\delta S_{\mathrm{l}}/\delta A$ and $\delta S_{\mathrm{l}}/\delta \phi^I$. This definition agrees with the beta function derived from the conformal Ward identity, i.e. the trace of the Brown-York stress tensor. Therefore, knowledge of $S_{\mathrm{l}}$ completely specifies the form of the beta function. Combined with the original hamiltonian constraint, this determines all the equations of motion, as we have shown above. Furthermore, when the space is $3+1$ Poincaré invariant, the local action $S_{\mathrm{l}}$ is determined by the vacuum energy density of the boundary field theory, which is proportional to the trace of the stress-energy tensor. We have derived an RG equation from gravity by directly computing the trace of the quasilocal stress tensor. The resulting beta function agrees exactly with the one discussed in [32]. The hamiltonian constraint (3.5) can be rewritten as (2.4) which, as we have seen (4.13), simply defines the flow of the c-function in our formalism.

We have shown that if the beta functions of a 4d field theory are given by gradients of a certain potential, then the RG flow can be expressed as a classical solution of 5-dimensional gravity coupled to scalar fields.[9] The Raychaudhuri equation in 5 dimensions automatically guaranteed a c-theorem for such flows. Equivalently, a 4d RG flow relates to 5d gravity when the beta functions are given by the gradient of a monotonic c-function (4.5), while the c-function is related to the trace of the stress tensor as in (4.4). Given the beta functions, stress tensor and c-function of a field theory we can always rewrite RG flow in terms of 5d gravity so long as the symmetric, positive metric $G^{IJ}$ in (4.5) can be defined. Similar results may be derived for RG flows of field theories in other dimensions. Famously, beta functions of *any* renormalizable theory in two dimensions can be expressed as gradients of a c-function [25] with a positive symmetric metric appearing in (4.5). It has been suggested that a holographic c-function appropriate to 2-dimensional theories will satisfy $c \sim 1/(a\langle T\rangle + U(\phi))$ [20]. Using this to define $U(\phi)$, and (2.4) to define $V(\phi)$, we expect that any such RG flow is expressible in terms of a gravity background.

What characterizes a theory which realizes gradient beta functions of the form discussed in this paper? By construction, the large-$N$, conformal theories appearing in the AdS/CFT correspondence have the requisite traits. However, the symmetries and properties guaranteeing (2.3) (or a monotonic c-function) have not been generally understood. Based on the AdS/CFT experience, we expect that a theory might have an RG flow satisfying (2.3) in some limit of parameters, but that deviations from the limit produce systematic corrections. These should be compared to modified spacetime equations of motion arising from higher derivative terms added to the action (3.1). In string theory, such terms originate in propagation of the excited states of string and in loop corrections.

Only some field theory RG schemes can be expected to have holographic descriptions in gravity. Within our analysis, there is a scheme dependence in the definition of the stress tensor $T$ and the potentials $U$ and $V$ in field theory. However, these stress tensor ambiguities can be precisely matched by modifications in the counterterm scheme (3.2) for the gravitational stress tensor.[10] More generally, different RG schemes involve different ways of imposing a cutoff and integrating out modes or, alternatively, different choices of counterterms. The AdS/CFT correspondence suggests that an appropriate class of RG schemes coarsens field variables by convolving them against a smearing kernel [33]. Even this cannot be entirely sufficient as

[9] By construction, the examples of holographic RG flows developed in [16, 22, 23, 24] all fit our formalism.

[10] One such ambiguity, affecting the definition of the trace anomaly on a curved manifold is discussed in [13].

the field theory approaches the deep infrared. In this limit, the gravity description involves a large bubble of essentially flat space at the center of an AdS spacetime. Since the field theory is in the deep infrared, it appears that the physics of homogeneous modes describes the flat space region. In the AdS/CFT case, these modes constitute the quantum mechanics of a large matrix, in an echo of the M(atrix) model of M-theory [34]. This suggests that recovery of the equations of motion of a flat space region in AdS requires implementation of a "matrix renormalization group" relating the physics of $\mathrm{SU}(N)$ to $\mathrm{SU}(N-k)$.

Holographic realizations of the renormalization group associate field theory scales with radial positions in a higher-dimensional spacetime. The full set of 5-dimensional diffeomorphisms can put "bumps" in surfaces at fixed radial positions. Recovering such transformations from 4-dimensional field theory will certainly involve a *local* notion of the renormalization group where the coarsening scale varies from point to point. This automatically requires consideration of theories with spatially varying couplings. It will be illuminating to elucidate the relation between local RG invariance in field theory and higher-dimensional diffeomorphism invariance.



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Recent work which overlaps with the content of this paper appears in [35].

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